On a dispersion problem in grid labeling

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Abstract

Given k labelings of a finite d-dimensional cubical grid, define the *combined distance* between two labels to be the sum of the ℓ_1-distance between the two labels in each labeling. We want to construct k labelings which maximize the minimum combined distance between any two labels. When $d = 1$, this can be interpreted as placing n non-attacking rooks in a k-dimensional chessboard of size n in such a way to maximize the minimum ℓ_1-distance between any two rooks. *Rook placements* are also known as *Latin Hypercube Designs* in the literature.

In this paper, we revisit this problem with a more geometric approach. Instead of providing explicit but complicated formulas, we construct rook placements in a k-dimensional chessboard of size n as certain lattice-like structures for certain well-chosen values of n. Then, we extend these constructions to any values of n using geometric arguments. With this method, we present a clean and geometric description of the known optimal rook placements in the 2-dimensional square grid. Furthermore, we provide asymptotically optimal constructions of k labelings of d-dimensional cubical grids which maximize the minimum combined distance.

Finally, we discuss the extension of this problem to labelings of an arbitrary graph. We prove that deciding whether a graph has two labelings with combined distance at least 3 is at least as hard as graph isomorphism.

1 Introduction

Let $L_1, \ldots, L_k$ be k bijections from the cells of a d-dimensional cubical grid of size n to a label set S of n^d symbols. Then each symbol in S labels k cells, one in each of the k labelings. Define the *combined distance* between two symbols x and y in S as the sum of the ℓ_1-distances between the two cells labeled by x and y in each labeling. How to arrange the symbols of the k labelings such that the minimum combined distance between any two symbols is maximized? We refer to Figure 1 for an example with $n = 3$ and $k = d = 2$.

This grid labeling problem was posed at the open problems session of CCCG 2009 [4] by Belén Palop, who formulated the problem from her research with Zhenghao Zhang in wireless communication. It has many applications to wireless communication, in particular, permutation code generation [7, Chapter 9]. A permutation code uses a grid of symbols for each channel when transmitting data over multiple channels; transmission errors are more easily detected if the combined distance between any pair of symbols in the grids is large.

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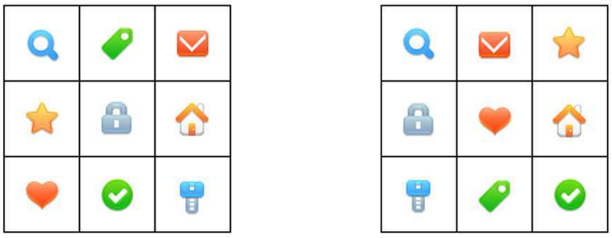

Figure 1: Two labelings of a 3×3 grid. With the first labeling fixed, the second labeling is one of 840 solutions for which the minimum combined distance is 3.

Throughout the paper, we denote by $\langle\, n \,\rangle$ the set $\{0, 1, 2, \ldots, n-1\}$. A *labeling* of the d-dimensional cubical grid of size n is a bijection $L : \langle\, n \,\rangle^d \to \langle\, n^d \,\rangle$ which assigns a *label* of $\langle\, n^d \,\rangle$ to each *grid cell* of $\langle\, n \,\rangle^d$. Given k labelings $L_1, \ldots, L_k$ of $\langle\, n \,\rangle^d$, we denote the *combined distance* between two labels $x, y \in \langle\, n^d \,\rangle$ by $\mathrm{CD}(L_1, \ldots, L_k, x, y) := \sum_{i=1}^{k} \|L_i^{-1}(x) - L_i^{-1}(y)\|_1$. Finally, we denote by

$$\gamma(k, n, d) := \max_{L_1, \ldots, L_k} \min_{x \neq y \in \langle\, n^d \,\rangle} \mathrm{CD}(L_1, \ldots, L_k, x, y)$$

the maximum value, among all k-tuples of labelings of the d-dimensional cubical grid of size n, of the minimal combined distance between any two distinct labels.

Assume for now that the dimension d of the grid is fixed to 1, that is, we investigate k-tuples $L_1, \ldots, L_k$ of labelings of the 1-dimensional array $\langle\, n \,\rangle$. Considering for each symbol x the point of $\mathbb{R}^k$ whose ith coordinate is the position of x in the ith labeling L_i, we obtain a set of n points in a k-dimensional grid of size n, such that no two points share a coordinate in any dimension. In other words, a set of n non-attacking rooks in a k-dimensional chessboard. We call such a configuration a *rook placement*. Moreover, the combined distance between two distinct labels in the k labelings of the 1-dimensional array is the ℓ_1-distance between the two corresponding rooks in the rook placement. Thus, $\gamma(k, n, 1)$ is precisely the maximal value, among all rook placements in a k-dimensional chessboard of size n, of the minimal ℓ_1-distance between two rooks.

Among results concerning optimal rook placements with respect to different ℓ_p-distances [2], van Dam et al. proved that $\gamma(2, n, 1) = \left\lfloor \sqrt{2n+2} \right\rfloor$. For k-dimensional chessboards, van Dam et al. [3] proved that the maximal value $\gamma(k, n, 1)$ is at most $\left\lfloor \frac{k}{3}(n+1) \right\rfloor$, but observed that this bound is certainly not optimal. In these two papers and more generally in the operation research literature, rook placements are referred as *Latin Hypercube Designs* [2, 3]. LHDs are useful in obtaining approximation models for black-box functions that may have too many combinations of input parameters and need to be tested on only a reduced subset of the combinations. For the sake of understanding, we will prefer the term rook placement rather than LHD in this article. A dynamic survey on related topics in graph labeling can be found in [5].

In this paper, we revisit the optimal rook placement problem with a more geometric approach. We first provide simple and explicit descriptions for optimal rook placements in a k-dimensional chessboard of size n, but only for certain values of n. For these values, our rook placements can be understood geometrically as lattice-like points sets. Then, we use these particular cases to generate good rook placements for arbitrary n. This approach enables us to focus on friendly values of n, and thus to avoid unnecessary technical calculations for general n. In particular, we present a clean and geometric description of the optimal rook placements presented in [2]. Furthermore, we obtain the following asymptotically tight bounds for the maximal value of the minimal ℓ_1-distance between two rooks of a rook placement in the k-dimensional chessboard of size n:

Theorem 1. *For any integers $k \geq 2$ and $n \geq 2$,*

$$k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} \leq \gamma(k, n, 1) \leq \frac{n-1}{(n/k!)^{1/k} - 1}.$$

With the same techniques, we then generalize these bounds to grid labelings of a d-dimensional cubical grid, for $d \geq 2$:

Theorem 2. *For any integers $k \geq 2$, $n \geq 2$, and $d \geq 1$,*

$$k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} \leq \gamma(k,n,d) \leq \frac{n-1}{(n^d/(dk)!)^{1/(dk)} - 1}.$$

In particular, $\gamma(k,n,d) = \Theta(n^{1-1/k})$ if k and d are constants.

Observe that our lower bounds, in conjunction with the upper bounds, yield a very simple $O(kn^d)$-time constant-factor approximation algorithm for the optimization problem of maximizing the combined distance of k labelings of a d-dimensional grid, for any constant k and d.

On the other hand, the problem becomes much more difficult to handle when generalized to the graph setting. Now let G be a graph of n vertices, and let S be a set of n symbols. Define a *labeling* of the graph G to be a bijection that assigns a distinct symbol in S to each vertex in G, and define the *distance* between two vertices in G as the number of edges in a shortest path between them. Then define the combined distance of multiple labelings of a graph in a similar way as that for a grid. We obtain the following theorem:

Theorem 3. *Deciding whether a graph has two labelings with combined distance at least 3 is at least as hard as graph isomorphism.*

The paper is organized as follows. In Section 2, we present constructions of labelings with large minimum combined distance. We present two purely combinatorial constructions for special values of n in Sections 2.1 and 2.2. Using the point of view of rook placements, we then reinterpret and extend these special labelings to arbitrary values of n in Section 2.3 and we obtain in particular Theorem 1. Section 2.4 generalizes these results to labelings of d-dimensional grids. Finally, we discuss in Section 3 the generalized problem of maximizing the minimal combined distance in graph labelings, and connect it to graph isomorphism to prove Theorem 3.

2 Constructions of labelings with large minimum combined distance

We fix an integer $k \geq 2$. The aim of this section is to present some combinatorial techniques to construct k labelings with a large minimum combined distance. We first focus on labelings of 1-dimensional arrays, and present two constructions which are interesting for different reasons:

A. Our first construction yields, for any integer m, a k-tuple of labelings of $\langle\, m^k \,\rangle$ with minimum combined distance $m^{k-1} - \frac{m^{k-1}-1}{m-1}$. Its interest lies in its simple combinatorial description.

B. Our second construction yields, for any integer m, a k-tuple of labelings of $\langle\, km^k \,\rangle$ with minimum combined distance km^{k-1}. It is our most efficient construction, and it is proved to be optimal when $k = 2$.

Our presentation of these two constructions only produces labelings of $\langle\, n \,\rangle$ for certain specific values of n. To treat all other values of n, we use the interpretation of k-tuples of labelings of $\langle\, n \,\rangle$ in terms of rook placements in the hypercube $\langle\, n \,\rangle^k$. In this setting, both our constructions can be thought of as the traces on $\langle\, n \,\rangle^k$ of lattice-like structures in $\mathbb{R}^k$, and a simple geometric construction extends these constructions to general values of n. For convenience, we will use L_0 as an alias for L_k in our descriptions of the two constructions in Sections 2.1 and 2.2, and refer to the k labelings as $L_0, \ldots, L_{k-1}$ instead of $L_1, \ldots, L_k$.

2.1 Construction A

We present our first construction only for $n = m^k$ and $m \geq 2$. Let $\phi : \langle\, m\,\rangle^k \to \langle\, m^k\,\rangle$ be the bijection defined as $\phi(x_{k-1}, \ldots, x_0) := \sum_{j=0}^{k-1} x_j m^j$. The reciprocal bijection ϕ^{-1} associates to an integer its decomposition in the m-ary number system, using k digits. Observe that we write the least significant digit to the right to be consistent with the usual conventions. Let $\sigma : \langle\, m\,\rangle^k \to \langle\, m\,\rangle^k$ be the cyclic permutation defined as $\sigma(x_{k-1}, \ldots, x_1, x_0) := (x_0, x_{k-1}, \ldots, x_1)$. For $0 \leq i \leq k-1$, we define a labeling A_i of $\langle\, m^k\,\rangle$ as

$$A_i := \phi \circ \sigma^i \circ \phi^{-1}.$$

In other words, the m-ary decompositions of a label and of its position in the labeling A_i are just cyclically permuted by σ^i. Observe that the inverse permutation of A_i is given by

$$A_i^{-1} = \phi \circ \sigma^{k-i} \circ \phi^{-1}.$$

Proposition 1. *The minimum combined distance of the k labelings $A_0, \ldots, A_{k-1}$ of $\langle\, m^k\,\rangle$ is bounded by*

$$\min_{x \neq y \in \langle\, m^k\,\rangle} \mathrm{CD}(A_0, \ldots, A_{k-1}, x, y) \geq m^{k-1} - \frac{m^{k-1} - 1}{m - 1}.$$

Proof. Observe first that for any two elements $(x_{k-1}, \ldots, x_0)$ and $(y_{k-1}, \ldots, y_0)$ of $\langle\, m\,\rangle^k$, the distance between the cells $\phi(x_{k-1}, \ldots, x_0)$ and $\phi(y_{k-1}, \ldots, y_0)$ in the array $\langle\, m^k\,\rangle$ is at least

$$|\phi(x_{k-1}, \ldots, x_0) - \phi(y_{k-1}, \ldots, y_0)| \geq m^{k-1}|x_{k-1} - y_{k-1}| - \sum_{j=0}^{k-2} m^j |x_j - y_j|.$$

Consequently, for any two distinct elements $(x_{k-1}, \ldots, x_0)$ and $(y_{k-1}, \ldots, y_0)$ of $\langle\, m\,\rangle^k$, the combined distance $\mathrm{CD}(A, x, y) := \mathrm{CD}(A_0, \ldots, A_{k-1}, x, y)$ between the labels $x := \phi(x_{k-1}, \ldots, x_0)$ and $y := \phi(y_{k-1}, \ldots, y_0)$ in the k labelings $A_0, \ldots, A_{k-1}$ is at least

$$\begin{aligned}
\mathrm{CD}(A, x, y) &= \sum_{i=0}^{k-1} \left|A_i^{-1}(x) - A_i^{-1}(y)\right| \\
&= \sum_{i=0}^{k-1} |\phi(x_{k-i-1}, \ldots, x_0, x_{k-1}, \ldots, x_{k-i}) - \phi(y_{k-i-1}, \ldots, y_0, y_{k-1}, \ldots, y_{k-i})| \\
&\geq \sum_{i=0}^{k-1} \left(m^{k-1}|x_{k-i-1} - y_{k-i-1}| - \sum_{j=0}^{k-2} m^j |x_{(j-i) \bmod k} - y_{(j-i) \bmod k}| \right) \\
&= \left(\sum_{i=0}^{k-1} |x_i - y_i| \right) \left(m^{k-1} - \sum_{j=0}^{k-2} m^j \right) \\
&\geq m^{k-1} - \frac{m^{k-1} - 1}{m - 1}.
\end{aligned}$$

□

Example 1. For $k = 2$ and $m = 4$, this construction yields the two labelings of $\langle\, 16\,\rangle$ with minimum combined distance 5 shown in Figure 2. For $k = 3$ and $m = 2$, this construction yields the three labelings of $\langle\, 8\,\rangle$ with minimum combined distance 6 shown in Figure 3. The numbers on top are the m-ary decompositions of the numbers in the array cells.

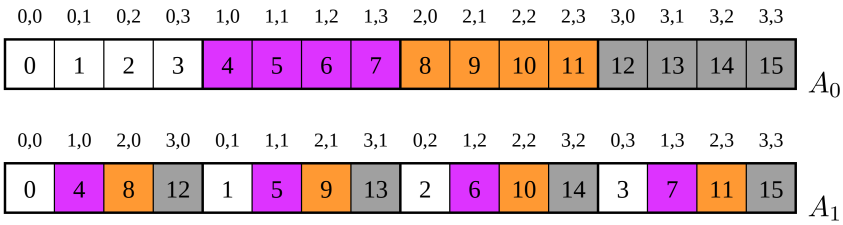

Figure 2: The two labelings A_0 and A_1 provided by construction A when $n = 16$, $k = 2$ and $m = 4$.

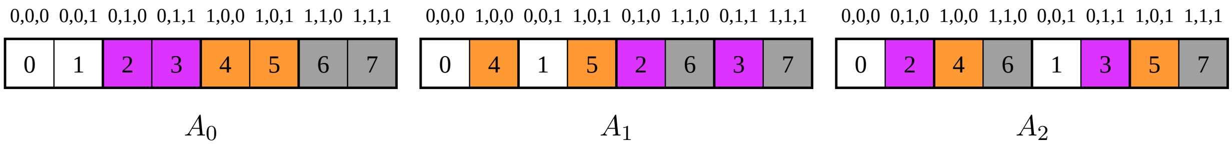

Figure 3: The three labelings A_0, A_1, and A_2 provided by construction A when $n = 8$, $k = 3$ and $m = 2$.

2.2 Construction B

We present our second construction only for $n = km^k$ and $m \geq 2$. For a fixed integer m we construct k labelings $B_0, \ldots, B_{k-1}$ of the array $\langle\, km^k \,\rangle$. To construct the labeling B_i, we first assign a color $\alpha_i(x)$ to each cell x of $\langle\, km^k \,\rangle$ such that

$$\alpha_i(x) := \left\lfloor \frac{x}{m^{i-1}} \right\rfloor \bmod m.$$

Intuitively, for $1 \leq i \leq k-1$, the cell x is colored by $\alpha_i(x)$ according to its ith least significant digit in its m-ary decomposition. Observe that the color $\alpha_0(x)$ is always equal to 0. The labeling B_i is then defined for all cells $x \in \langle\, km^k \,\rangle$ by

$$B_i(x) := \Big(x - km^{k-1}\alpha_i(x)\Big) \bmod km^k.$$

In other words, for all $0 \leq p \leq m-1$, the labeling B_i cyclically permutes the set of all cells x with color $\alpha_i(x) = p$, and the amplitude of this permutation is proportional to p. In particular, we have $\alpha_i(x) = \alpha_i(B_i(x))$ and it is easy to describe the inverse permutation of B_i for all labels $x \in \langle\, km^k \,\rangle$ as

$$B_i^{-1}(x) = \Big(x + km^{k-1}\alpha_i(x)\Big) \bmod km^k.$$

Note that B_0 is the identity permutation since $\alpha_0(x) = 0$ for all $x \in \langle\, km^k \,\rangle$.

Proposition 2. *The minimum combined distance of the k labelings $B_0, \ldots, B_{k-1}$ of $\langle\, km^k \,\rangle$ is bounded by*

$$\min_{x \neq y \in \langle\, km^k \,\rangle} \textsc{cd}(B_0, \ldots, B_{k-1}, x, y) \geq km^{k-1}.$$

Proof. Let x and y be two distinct labels of $\langle\, km^k \,\rangle$. For $0 \leq i \leq k-1$, write

$$B_i^{-1}(x) = x + km^{k-1}\alpha_i(x) + r_i km^k$$
$$\text{and} \quad B_i^{-1}(y) = y + km^{k-1}\alpha_i(y) + s_i km^k$$

for some integers r_i and s_i. We consider two cases:

(1) If $\alpha_i(x) = \alpha_i(y)$ for all i, then $x - y$ is a non-zero multiple of m^{k-1}. Thus, for all i, the difference $B_i^{-1}(x) - B_i^{-1}(y) = x - y + (r_i - s_i)km^k$ is also a non-zero multiple of m^{k-1}, and

$$\textsc{cd}(B_0, \ldots, B_{k-1}, x, y) = \sum_{i=0}^{k-1} \left|B_i^{-1}(x) - B_i^{-1}(y)\right| \geq km^{k-1}.$$

(2) Otherwise, $\alpha_j(x) \neq \alpha_j(y)$ for some $j \neq 0$. Then

$$
\begin{aligned}
\text{CD}(B_0,\dots,B_{k-1},x,y) &= \sum_{i=0}^{k-1} \left|B_i^{-1}(x) - B_i^{-1}(y)\right| \;\geq\; |B_j^{-1}(x) - B_j^{-1}(y)| + |B_0^{-1}(x) - B_0^{-1}(y)| \\
&= |B_j^{-1}(x) - B_j^{-1}(y)| + |x-y| \;\geq\; |B_j^{-1}(x) - B_j^{-1}(y) - x + y| \\
&= km^{k-1}|\alpha_j(x) - \alpha_j(y) + (r_j - s_j)m| \;\geq\; km^{k-1}.
\end{aligned}
$$

The last inequality holds since $1 \leq |\alpha_j(x) - \alpha_j(y)| \leq m-1$. $\square$

Example 2. For $k = 2$ and $m = 3$, this construction yields the two labelings of $\langle\, 18\,\rangle$ with minimum combined distance 6 shown in Figure 4. For $k = 3$ and $m = 2$, this construction yields the three labelings of $\langle\, 24\,\rangle$ with minimum combined distance 12 shown in Figure 5. The numbers on top are the three least significant digits of the m-ary decompositions of the array cell indices.

	0,0,0	0,0,1	0,0,2	0,1,0	0,1,1	0,1,2	0,2,0	0,2,1	0,2,2	1,0,0	1,0,1	1,0,2	1,1,0	1,1,1	1,1,2	1,2,0	1,2,1	1,2,2
B_0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
B_1	0	13	8	3	16	11	6	1	14	9	4	17	12	7	2	15	10	5

Figure 4: The two labelings B_0 and B_1 provided by construction B when $n = 18$, $k = 2$ and $m = 3$.

	0,0,0	0,0,1	0,1,0	0,1,1	1,0,0	1,0,1	1,1,0	1,1,1	0,0,0	0,0,1	0,1,0	0,1,1	1,0,0	1,0,1	1,1,0	1,1,1	0,0,0	0,0,1	0,1,0	0,1,1	1,0,0	1,0,1	1,1,0	1,1,1
B_0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
B_1	0	13	2	15	4	17	6	19	8	21	10	23	12	1	14	3	16	5	18	7	20	9	22	11
B_2	0	1	14	15	4	5	18	19	8	9	22	23	12	13	2	3	16	17	6	7	20	21	10	11

Figure 5: The three labelings B_0, B_1, and B_2 provided by construction B when $n = 24$, $k = 3$ and $m = 2$.

Remark 1. Both constructions A and B can be generalized to arbitrary n. In the next subsection we present a unified view of the two constructions and provide a conceptually simple meta-method for such generalizations.

2.3 Rook placements

In this section, we interpret the minimum combined distance of k labelings of a 1-dimensional array $\langle\, n\,\rangle$ as the minimum distance in a rook placement in the k-dimensional hypercube $\langle\, n\,\rangle^k$. Let us first state a precise definition:

Definition 1. *A* (k,n)-rook placement *is a subset R of the k-dimensional hypercube $\langle\, n\,\rangle^k$ with precisely one element in the subspace $\langle\, n\,\rangle^{p-1} \times \{q\} \times \langle\, n\,\rangle^{k-p}$ for each $1 \leq p \leq k$ and $0 \leq q \leq n-1$.*

In other words, a (k,n)-rook placement is a maximal set of non-attacking rooks in $\langle\, n\,\rangle^k$, where a rook positioned in $(x_1,\dots,x_k)$ can attack the subspaces $\langle\, n\,\rangle^{p-1} \times \{x_p\} \times \langle\, n\,\rangle^{k-p}$ for $1 \leq p \leq k$ (see Figure 6).

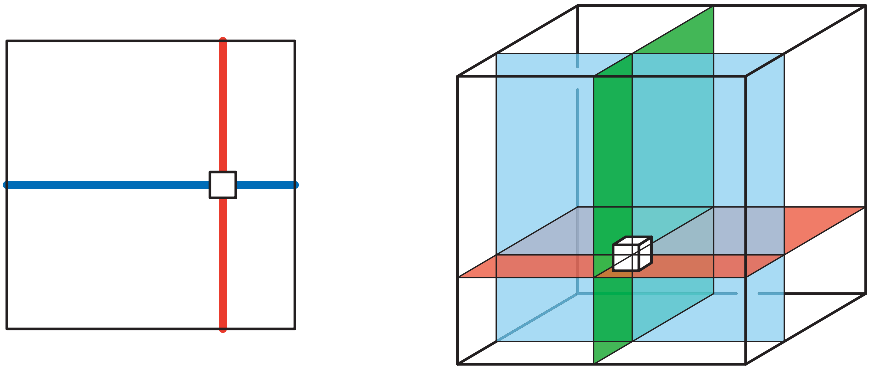

Figure 6: The affine spaces a rook can attack.

There is an immediate correspondence between k-tuples of labelings of the 1-dimensional array $\langle\, n \,\rangle$ and (k, n)-rook placements:

- given k labelings $L_1, \dots, L_k$ of $\langle\, n \,\rangle$, the subset $R(L_1, \dots, L_k) := \left\{ (L_1^{-1}(x), \dots, L_k^{-1}(x)) \;\middle|\; x \in \langle\, n \,\rangle \right\}$ of $\langle\, n \,\rangle^k$ is a (k, n)-rook placement;
- reciprocally, a (k, n)-rook placement R has n rooks, whose pth coordinates are all distinct (for any fixed $1 \le p \le k$). If we arbitrarily label the rooks of R from 0 to $n - 1$, the order of the rooks according to their pth coordinate defines a labeling $L_p(R)$ of $\langle\, n \,\rangle$.

Observe that we do not change the rook placement when permuting the labels of its rooks: for any permutations $L_1, \dots, L_k$ and τ of $\langle\, n \,\rangle$, we have $R(\tau \circ L_1, \dots, \tau \circ L_k) = R(L_1, \dots, L_k)$. We can therefore assume that L_1 is the identity permutation. Consequently, the number of (k, n)-rook placements is $(n!)^{k-1}$.

Furthermore, the above correspondence between k-tuples of labelings of $\langle\, n \,\rangle$ and (k, n)-rook placements preserves metric properties: the combined distance between two labels x and y in k labelings $L_1, \dots, L_k$ of $\langle\, n \,\rangle$ is precisely the ℓ_1-distance between the two corresponding rooks $(L_1^{-1}(x), \dots, L_k^{-1}(x))$ and $(L_1^{-1}(y), \dots, L_k^{-1}(y))$ in the (k, n)-rook placement $R(L_1, \dots, L_k)$. We call *minimum distance* of a finite point set S of $\mathbb{R}^k$ the minimum pairwise ℓ_1-distance between two points of S.

To illustrate the interest of this geometric point of view, let us first prove the upper bound of Theorem 1:

Lemma 1. *For any integers $k \ge 2$ and $n \ge 2$,*

$$\gamma(k, n, 1) \le \frac{n - 1}{(n/k!)^{1/k} - 1}.$$

Proof. We prove the result in the setting of rook placements by a simple volume argument. Consider a (k, n)-rook placement R, and let δ be the minimum distance between two rooks of R. Then the ℓ_1-balls of radius $\delta/2$ centered at the rooks of R are disjoint and contained in the cube $[-\delta/2, n - 1 + \delta/2]^k$. Since each ball has volume $\delta^k/k!$, this yields the inequality $n\delta^k/k! \le (n - 1 + \delta)^k$, and thus the upper bound of the lemma. □

To prove the lower bound of Theorem 1, we will use more general configurations of integer points in $\mathbb{R}^k$ to obtain (k, n)-rook placements with large minimum distance, for all values of n. The principal ingredient of our constructions is the following proposition:

Proposition 3. *If there exists a set of n integer points in $\mathbb{Z}^k$ with minimum distance δ such that the projection of these points on each axis is an interval of consecutive integers (with possible repetitions), then there exists a (k, n)-rook placement with minimum distance δ.*

Proof. Let S be such a set of n integers. We label the points of S arbitrarily from 0 to $n-1$. For each direction i, we then construct a labeling L_i of $\langle\, n \,\rangle$ which respects the order of the ith coordinate of the points of S, and where points with equal ith coordinate are ordered arbitrarily. Since the projection of S in each direction covered an interval of integers, the distance between two points in each direction can only increase during this construction, and the minimum distance of the (k,n)-rook placement $R(L_1,\ldots,L_k)$ is at least that of S. $\square$

A simple way to obtain such point sets S on which we can easily control the minimum distance is to use lattices of $\mathbb{R}^k$. Remember that a *lattice* of $\mathbb{R}^k$ is the set of integer linear combinations of k linearly independent vectors of $\mathbb{R}^k$; see [6, Chapter 1]. We call a *(k,n)-rook lattice* any sublattice L of the integer lattice $\mathbb{Z}^k$ whose trace $L \cap \langle\, n \,\rangle^k$ on the hypercube $\langle\, n \,\rangle^k$ is a (k,n)-rook placement and which contains ne_0 (e_0 is the first vector of the canonical basis of $\mathbb{R}^k$). Applying Proposition 3, a good (k,ν)-rook lattice provides good (k,n)-rook placements not only for $n=\nu$, but for any larger value of n:

Proposition 4. *If there exists a (k,ν)-rook lattice with minimum distance δ, then there exists a (k,n)-rook placement with minimum distance δ for all $n \geq \nu - 1$.*

Proof. Let L be a (k,ν)-rook lattice of minimum distance δ. For $n=\nu-1$, consider the point configuration $L \cap \{1,\ldots,\nu-1\}^k$: it has minimum distance δ and projects bijectively on $\{1,\ldots,\nu-1\}$ in each direction. For $n \geq \nu$, consider the trace of L on $\langle\, n \,\rangle \times \langle\, \nu \,\rangle^{k-1}$. Since $ne_0 \in L$, this trace projects bijectively on $\langle\, n \,\rangle$ in the first direction and surjectively on $\langle\, \nu \,\rangle$ in all the other directions. The result thus follows from Proposition 3. $\square$

In the remaining of this section, we first use this result to reinterpret van Dam et al.'s rook placements in the square [2] in a neat and geometric way. Our description provides the same rook placements and avoids tedious and technical calculations. We then apply Proposition 4 to extend the constructions of Sections 2.1 and 2.2 to any value of n.

Example 3 (Rook placements in the square)**.** We consider two families of lattices in the plane (see Figure 7):

(a) The lattice generated by (m,m) and $(1,2m+1)$ is a $(2,2m^2)$-rook lattice with minimum distance $2m$.

(b) The lattice generated by $(m+1,m)$ and $(1,2m+1)$ is a $(2,2m^2+2m+1)$-rook lattice with minimum distance $2m+1$.

Note that the trace of these rook lattices on their corresponding square gives precisely the rook placements of [2] (up to a reflection with respect to the vertical axis).

From these two families and using Proposition 4, we obtain in a much simpler way the lower bound on $\gamma(2,n,1)$ in [2]:

Proposition 5. *For any integer n, $\gamma(2,n,1) \geq \left\lfloor \sqrt{2n+2} \right\rfloor$.*

Proof. Let m be any integer. Since there exists a $(2,2m^2)$-rook lattice with minimum distance $2m$, Proposition 4 implies for any integer n with $2m^2-1 \leq n \leq 2m^2+2m-1$ that $\left\lfloor \sqrt{2n+2} \right\rfloor = 2m \leq \gamma(2,n,1)$. Similarly, since there exists a $(2,2m^2+2m+1)$-rook lattice with minimum distance $2m+1$, Proposition 4 implies for any integer n with $2m^2+2m \leq n \leq 2m^2+4m$ that $\left\lfloor \sqrt{2n+2} \right\rfloor = 2m+1 \leq \gamma(2,n,1)$. $\square$

We have seen in Lemma 1 that $\gamma(2,n,1)$ is bounded by $(n-1)/(\sqrt{n/2}-1)$. Together with Proposition 5, this implies that $\gamma(2,n,1) \sim \sqrt{2n}$. In fact, using a similar but slightly refined packing argument as in our proof of Lemma 1, van Dam et al. [2] proved that the bound in Proposition 5 is in fact the exact value of $\gamma(2,n,1)$:

$$\gamma(2,n,1) = \left\lfloor \sqrt{2n+2} \right\rfloor.$$

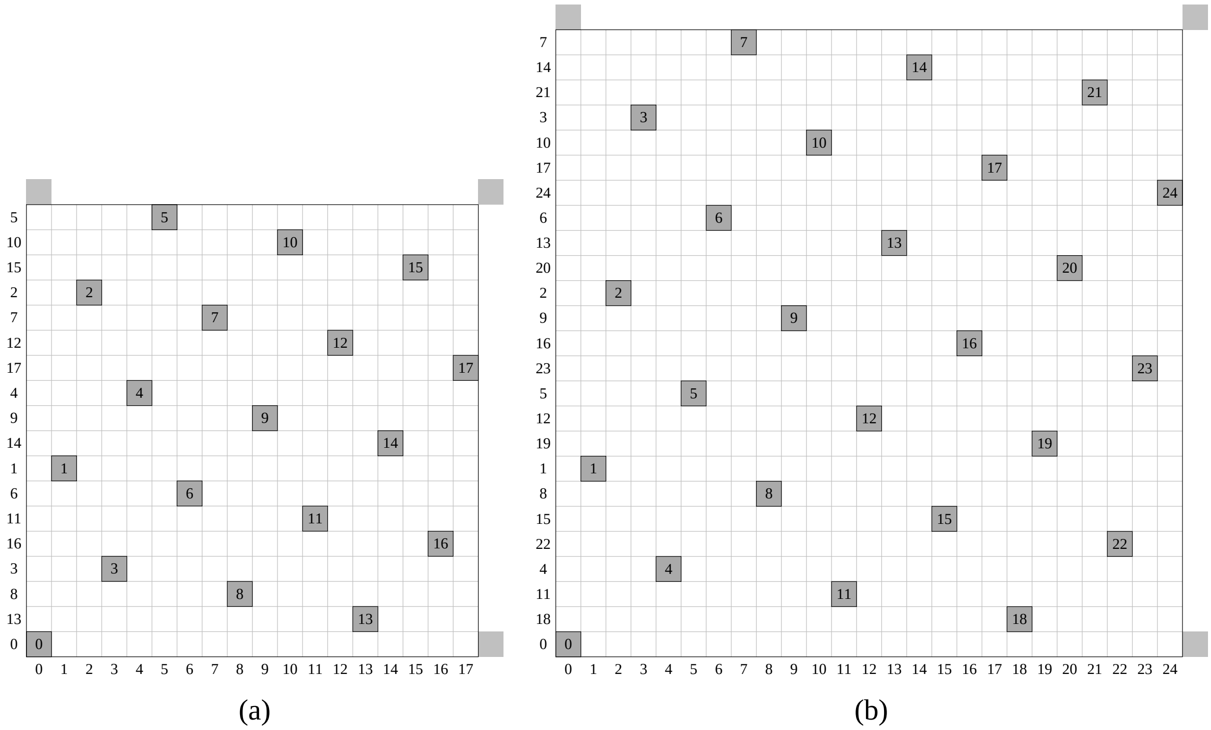

Figure 7: Examples of two optimal families of rook lattices in the square. (a) Lattice generated by the vectors (m, m) and $(1, 2m + 1)$, for $m = 3$. (b) Lattice generated by the vectors $(m + 1, m)$ and $(1, 2m + 1)$, for $m = 3$.

Example 4 (Construction A, revisited)**.** Denote by $(e_0, \ldots, e_{k-1})$ the canonical basis of $\mathbb{R}^k$. Consider the lattice $U(k, m)$ of $\mathbb{R}^k$ generated by the vectors $u_j := \sum_{i=0}^{k-1} m^{(j+i) \bmod k} e_i$, for $0 \leq j \leq k - 1$. In other words, the matrix whose column vectors are $u_0, \ldots, u_{k-1}$ is a circulant matrix $M(k, m)$ whose first row is $(1, m, \ldots, m^{k-1})$. See Figure 8 for an example.

Lemma 2. *The (k, m^k)-rook placement $R(A_0, \ldots, A_{k-1})$ is formed by the points of $U(k, m)$ located in the hypercube $\langle\, m^k - 1\,\rangle^k$ together with the point $(m^k - 1)\sum_{i=0}^{k-1} e_i$.*

Proof. For any $x := \phi(x_{k-1}, \ldots, x_0) \in \langle\, m^k\,\rangle$, the rook labeled by x in $R(A_0, \ldots, A_{k-1})$ is positioned at

$$\sum_{i=0}^{k-1} A_i^{-1}(x) e_i = \sum_{i=0}^{k-1} \left(\sum_{\ell=0}^{k-1} x_{(\ell-i) \bmod k}\, m^{\ell} \right) e_i = \sum_{i=0}^{k-1} \left(\sum_{j=0}^{k-1} x_j\, m^{(j+i) \bmod k} \right) e_i$$
$$= \sum_{j=0}^{k-1} x_j \left(\sum_{i=0}^{k-1} m^{(j+i) \bmod k} e_i \right) = \sum_{j=0}^{k-1} x_j u_j,$$

and thus is an element of the lattice $U(k, m)$. For any i, we have $0 \leq A_i^{-1}(x) \leq m^k - 1$ and the last inequality is an equality if and only if $x = m^k - 1 = \phi(m - 1, m - 1, \ldots, m - 1)$. Thus, the rook labeled by x is either in $U(k, m) \cap \langle\, m^k - 1\,\rangle^k$, or equals $(m^k - 1)\sum_{i=0}^{k-1} e_i$.

For the reverse inclusion, we use a volume argument. Define the shifted hypercube $C := \left[-\frac{1}{2}, m^k - \frac{3}{2}\right]^k$ and the corresponding tiling $\mathcal{T} := C + \sum_{i=0}^{k-1} \mathbb{Z}\,(m^k - 1)e_i$ of the space $\mathbb{R}^k$. By inversion of the circulant matrix $M(k, m)$, the vector $(m^k - 1)e_i = m u_{(i-1) \bmod k} - u_i$ is in the lattice $U(k, m)$ for all $0 \leq i \leq k - 1$. Consequently, any tile of $\mathcal{T}$ contains the same number of points of the lattice $U(k, m)$. Since the boundary of C contains no point of $U(k, m)$, it follows that the number of points in C is the quotient of its volume by the

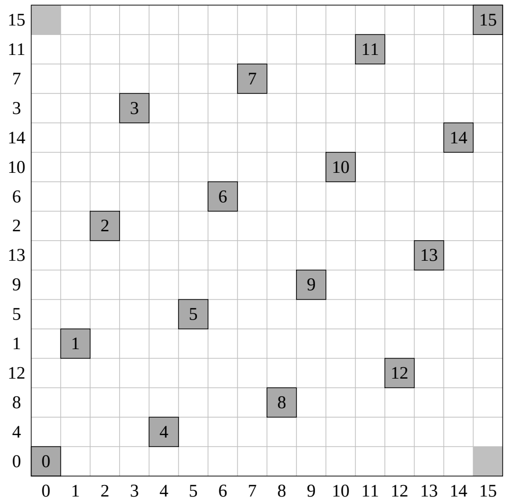

Figure 8: The lattice corresponding to the example in Figure 2 of construction A, for $n = 16$, $k = 2$ and $m = 4$.

volume of the fundamental parallelepiped of the lattice $U(k, m)$. The former clearly equals $(m^k - 1)^k$ while the later is the determinant of the circulant matrix $M(k, m)$, that is, $(m^k - 1)^{k-1}$. Consequently, the lattice $U(k, m)$ has precisely $m^k - 1$ points in C, hence in $\langle\, m^k - 1 \,\rangle^k$. This implies the reverse inclusion. $\square$

In other words, $U(k, m)$ is a (k, m^k)-rook lattice whose minimum distance is at least $m^{k-1} - \frac{m^{k-1}-1}{m-1}$. Applying Proposition 4, we obtain that for any $n \in \mathbb{N}$,

$$\gamma(k, n, 1) \geq \left\lfloor n^{1/k} \right\rfloor^{k-1} - \frac{\left\lfloor n^{1/k} \right\rfloor^{k-1} - 1}{\left\lfloor n^{1/k} \right\rfloor - 1}.$$

Example 5 (Construction B, revisited)**.** We finish by reinterpreting our Construction B in terms of rook lattices. When $k = 2$, the rook placement $R(B_0, B_1)$ is precisely the trace of the rook lattice generated by (m, m) and $(1, 2m + 1)$ which we saw in Example 3(a) (see also Figure 7(a)). As discussed previously, this rook lattice provides optimal rook placements in the square.

For $k \geq 3$, the (k, km^k)-rook placement $R(B_0, \ldots, B_{k-1})$ produced by construction B is not the trace of a lattice on $\langle\, km^k \,\rangle$. However, it is still sufficiently regular to apply Proposition 3.

Lemma 3. *For any integers $k \geq 2$ and $n \geq 2$,*

$$\gamma(k, n, 1) \geq k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1}.$$

Proof. Let $m := \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor$. Let S denote the set obtained by translations of the (k, km^k)-rook placement $R(B_0, \ldots, B_{k-1})$ by any integer multiple of $km^k e_0$. In other words, since B_0 is the identity permutation,

$$S = \left\{ (x, B_1^{-1}(x), \ldots, B_{k-1}^{-1}(x)) \;\middle|\; x \in \mathbb{Z} \right\}.$$

The trace of S on $\langle\, n \,\rangle \times \langle\, km^k \,\rangle^{k-1}$ projects bijectively on $\langle\, n \,\rangle$ on the first coordinate and surjectively on $\langle\, km^k \,\rangle$ on all other coordinates. A similar analysis as in the proof of Proposition 2 ensures that the minimum distance of S, like the minimum distance of $R(B_0, \ldots, B_{k-1})$, is at least km^{k-1} too. According to Propositions 2 and 3, we obtain a (k, n)-rook placement whose minimum distance is at least km^{k-1}. Thus,

$$\gamma(k, n, 1) \geq km^{k-1} = k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1}. \qquad \square$$

To summarize, Lemmas 1 and 3 prove Theorem 1 announced in the introduction:

Theorem 1. *For any integers $k \geq 2$ and $n \geq 2$, the maximal value of the minimum ℓ_1-distance between two rooks of a rook placement in a k-dimensional chessboard of size n is bounded by*

$$k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} \leq \gamma(k,n,1) \leq \frac{n-1}{(n/k!)^{1/k} - 1}.$$

2.4 Labelings of d-dimensional grids

We now extend our results to general dimension d, proving Theorem 2 announced in the introduction:

Theorem 2. *For any integers $k \geq 2$, $n \geq 2$, and $d \geq 1$, the maximal value of the minimum combined distance between any two labels for a k-tuple of labelings of the d-dimensional grid of size n is bounded by*

$$k \left\lfloor \left(\frac{n}{k}\right)^{1/k} \right\rfloor^{k-1} \leq \gamma(k,n,d) \leq \frac{n-1}{(n^d/(dk)!)^{1/(dk)} - 1}.$$

In particular, $\gamma(k,n,d) = \Theta(n^{1-1/k})$ if k and d are constants.

To generalize the lower bound from a one-dimensional array to a d-dimensional grid, we simply treat the d dimensions independently. The movement of a symbol in the $k-1$ labelings $L_1, \ldots, L_{k-1}$ in each direction depends only on the location of the symbol in the labeling L_0 in that particular direction, as described in the previous Sections. Thus we obtain a lower bound for the d-dimensional grid that is exactly the same as the lower bound for the one-dimensional array.

Example 6. For $k = 2$, $n = 8$, and $d = 2$, construction B yields the two labelings with minimum combined distance 4 shown in Figure 9.

0,7	1,7	2,7	3,7	4,7	5,7	6,7	7,7
0,6	1,6	2,6	3,6	4,6	5,6	6,6	7,6
0,5	1,5	2,5	3,5	4,5	5,5	6,5	7,5
0,4	1,4	2,4	3,4	4,4	5,4	6,4	7,4
0,3	1,3	2,3	3,3	4,3	5,3	6,3	7,3
0,2	1,2	2,2	3,2	4,2	5,2	6,2	7,2
0,1	1,1	2,1	3,1	4,1	5,1	6,1	7,1
0,0	1,0	2,0	3,0	4,0	5,0	6,0	7,0

L_0

0,3	5,3	2,3	7,3	4,3	1,3	6,3	3,3
0,6	5,6	2,6	7,6	4,6	1,6	6,6	3,6
0,1	5,1	2,1	7,1	4,1	1,1	6,1	3,1
0,4	5,4	2,4	7,4	4,4	1,4	6,4	3,4
0,7	5,7	2,7	7,7	4,7	1,7	6,7	3,7
0,2	5,2	2,2	7,2	4,2	1,2	6,2	3,2
0,5	5,5	2,5	7,5	4,5	1,5	6,5	3,5
0,0	5,0	2,0	7,0	4,0	1,0	6,0	3,0

L_1

Figure 9: Two labelings L_0 and L_1 of a square grid, obtained by construction B. For convenience, in this example we label each direction independently by using $\langle\, n \,\rangle^d$ labels, instead of $\langle\, n^d \,\rangle$ labels.

In turn, the upper bound for general d is obtained by an adapted packing argument. As in the case when $d = 1$, we can represent k labelings $L_1, \ldots, L_k$ of a d-dimensional grid $\langle\, n \,\rangle^d$ by the point configuration

$R(L_1,\ldots,L_k) := \left\{(L_1^{-1}(x),\ldots,L_k^{-1}(x)) \mid x \in \langle\, n\,\rangle^d\right\}$ of $(\langle\, n\,\rangle^d)^k \simeq \langle\, n\,\rangle^{dk}$. The combined distance between two labels $x, y \in \langle\, n^d\,\rangle$ is given by the ℓ_1-distance of the corresponding rooks $(L_1^{-1}(x),\ldots,L_k^{-1}(x))$ and $(L_1^{-1}(y),\ldots,L_k^{-1}(y))$ of $R(L_1,\ldots,L_k)$. Consequently, if $L_1,\ldots,L_k$ are k labelings of $\langle\, n\,\rangle^d$ with minimum combined distance δ, then the ℓ_1-balls of radius $\delta/2$ centered at the rooks of $R(L_1,\ldots,L_k)$ are disjoint and contained in the hypercube $[-\delta/2, n-1+\delta/2]^{dk}$. Since each of these balls has volume $\delta^{dk}/(dk)!$, this yields the inequality $n^d\delta^{dk}/(dk)! \le (n-1+\delta)^{dk}$, and thus the upper bound of Theorem 2.

3 Connection to graph isomorphism

In this section, we discuss the generalization of this problem to labelings of arbitrary graphs. Let G be a graph on n vertices, and let S be a set of n symbols. Define a *labeling* of the graph G as a bijection that assigns a distinct symbol in S to each vertex in G, and define the *distance* between two vertices in G as the number of edges in a shortest path between them. The *combined distance* between two labels $x, y \in S$ of the k labelings $L_1,\ldots,L_k$ of G is again defined as the sum of the distances in G of the vertices labeled by x and y in each labeling.

We first prove the following lemma:

Lemma 4. *A graph has two labelings with combined distance at least* 3 *if and only if the graph is a subgraph of its complement.*

Proof. We first prove the direct implication. Suppose that a graph G has two labelings L_1 and L_2 with combined distance at least 3. Then any two symbols assigned by one labeling to two adjacent vertices in G must be assigned by the other labeling to two non-adjacent vertices in G. That is, any two symbols assigned by one labeling to two adjacent vertices in G must be assigned by the other labeling to two adjacent vertices in the complement G' of G. Thus the two labelings L_1 and L_2 specify a bijection f from the vertices of G to the vertices of G' such that two vertices u and v are adjacent in G only if the corresponding two vertices $f(u)$ and $f(v)$ are adjacent in G'. Therefore G is a subgraph of its complement G'.

We next prove the reverse implication. Suppose G is a subgraph of its complement G'. Let f be a bijection from the vertices of G to the vertices of G' such that two vertices u and v are adjacent in G only if the corresponding two vertices $f(u)$ and $f(v)$ are adjacent in G'. Then in the graph G, two vertices u and v are adjacent only if the two vertices $f(u)$ and $f(v)$ are non-adjacent. Let L_1 and L_2 be two labelings of G such that the symbol assigned to a vertex v by L_1 is the same as the symbol assigned to the corresponding vertex $f(v)$ by L_2. Then the combined distance of the two labelings L_1 and L_2 is at least 3. $\square$

The problem *graph isomorphism* is that of deciding whether two graphs are isomorphic. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection f from V_1 to V_2 such that any two vertices u and v are adjacent in G_1 if and only if the corresponding two vertices $f(u)$ and $f(v)$ are adjacent in G_2. A graph is *self-complementary* if it is isomorphic to its complement. It is known that self-complementary graph recognition is polynomial-time equivalent to graph isomorphism [1]. Observe that a graph is isomorphic to its complement if and only if

(1) the graph is a subgraph of its complement, and

(2) the graph and its complement have the same number of edges.

Condition (2) can be easily checked in linear time. Together with Lemma 4, this completes the proof of Theorem 3:

Theorem 3. *Deciding whether a graph has two labelings with combined distance at least* 3 *is at least as hard as graph isomorphism.*

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